Limit of functions

Def: Let $A \subset \mathbb{R}$. A real number c is called a cluster point of A if $\forall \delta > 0, \exists x \in A$ s.t.

$0 < |x - c| < \delta$.

Def: ($\varepsilon - \delta$ definition)

Let $A \subset \mathbb{R}$ & c be a cluster point of A. $f: A \to \mathbb{R}$. Then $L \in \mathbb{R}$ is called the limit of f at c, (We say $\lim_{x \to c} f(x) = L$) if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$ with $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$.

Example 1: Use ε-δ def. to prove $\lim_{x\to 2}\frac{x+6}{x^2-2}=4$.

Proof: Clearly $f(x):=\frac{x+6}{x^2-2}$ has natural domain $\mathbb{R}\setminus\{\pm\sqrt{2}\}$. Hence 2 is a cluster point.

Note that

$\forall x\in\mathbb{R}\setminus\{\pm\sqrt{2}\}$,

$$|f(x)-4| = \left|\frac{x+6}{x^2-2}-4\right| = \frac{|4x+7|}{|x^2-2|}\cdot|x-2|.$$

If $|x-2|<\frac{1}{2}$, then $\frac{3}{2}<x<\frac{5}{2}$

$$\Rightarrow \frac{1}{4}<x^2-2<\frac{17}{4} \ \& \ |4x+7|<20.$$

Let $\varepsilon>0$ be given. Take $\delta:=\min\{\frac{1}{2},\frac{\varepsilon}{80}\}$.

Hence if $0<|x-2|<\delta$, then

$$|f(x)-4| = \frac{|4x+7|}{|x^2-2|}\cdot|x-2| < \frac{20}{\frac{1}{4}}\times\frac{\varepsilon}{80} = \varepsilon.$$

Therefore, $\lim_{x\to 2}\frac{x+6}{x^2-2}=4$.

☆

Example 2: Use ε-δ def. to prove $\lim_{x\to 4} \frac{4-x}{2-\sqrt{|x|}}$.

Proof: For $x \in [0,\infty)\setminus\{4\}$, we have

$$\frac{4-x}{2-\sqrt{|x|}} = \frac{4-x}{2-\sqrt{x}} = \frac{(2+\sqrt{x})(2-\sqrt{x})}{2-\sqrt{x}} = 2+\sqrt{x} \quad \&$$

$$\left|\frac{4-x}{2-\sqrt{|x|}} - 4\right| = |\sqrt{x}-2| = \frac{|x-4|}{\sqrt{x}+2} \le \frac{|x-4|}{2}.$$

Let $\varepsilon > 0$ be given. Set $\delta = \min\{\varepsilon, 1\} > 0$.

Now if $0 < |x-4| < \delta$, we have

$$\left|\frac{4-x}{2-\sqrt{|x|}} - 4\right| \le \frac{|x-4|}{2} < \frac{\delta}{2} < \varepsilon.$$

Therefore,
$$\lim_{x\to 4} \frac{4-x}{2-\sqrt{|x|}} = 4.$$

$\square$

Continuous Function

Def: Let $A \subset \mathbb{R}$, $c \in A$ & $f: A \to \mathbb{R}$.

We say f is continuous at c if $\forall \varepsilon > 0$,

$\exists \delta > 0$ s.t. for all $x \in A$ & $|x - c| < \delta$, we have

$|f(x) - f(c)| < \varepsilon$.

- Most of elementary functions we learn are continuous on the natural domains.

Examples:

- Constant : $f(x) \equiv c$.
- Identity : $f(x) = x$.
- Power : $f(x) = x^n$.

(· Trigonometric : $f(x) = \sin x$.)

Ex: Prove $f(x) = \frac{1}{x}$ is continuous on $\mathbb{R} \setminus \{0\}$.

Proof: Note: $\forall x, c \neq 0$, we have

$$|f(x) - f(c)| = \left|\frac{1}{x} - \frac{1}{c}\right| = \frac{1}{|x| \cdot |c|} \cdot |x-c|.$$

Suppose $c > 0$.

If $|x-c| < \frac{c}{2}$, then $0 < \frac{c}{2} < x < \frac{3c}{2}$.

In this case, $\frac{1}{|x|\cdot|c|}|x-c| \leq \frac{2}{c^2}|x-c|$.

Let $\varepsilon > 0$. Choose $\delta = \min\{\frac{c}{2}, \frac{c^2}{2}\varepsilon\}$.

Whenever $|x-c| < \delta$, we have

$$|f(x) - f(c)| = \frac{1}{|x|\cdot|c|}|x-c| < \frac{2}{c^2}|x-c| < \varepsilon.$$

(Check the case $c<0$ by yourself!)

Therefore, $\lim_{x \to c} f(x) = \frac{1}{c}$.